Linear Algebra I

17/12/2018, Monday, 15:00 – 17:00

1 $(5+10+5+5=25 \text{ pts})$ **Linear systems of equations**

A company produces 3 types of products: **A**, **B**, and **C**. For a type **A** product, it takes 20 minutes to assemble, 4 minutes to test, and 4 minutes to pack. A type **B** product requires 24 minutes to assemble, 5 minutes to test, and 4 minutes to pack. Finally, a type **C** product requires 12 minutes to assemble, 3 minutes to test, and 3 minutes to pack. Given that the company can afford 3120 minutes per day for assembling, 680 minutes for testing, and 640 minutes for packing, we want to find how many of each kind can be produced in a day.

(a) Let x, y, and z be number of, respectively, type **A**, **B**, and **C**, products that produced each day. Find the linear equations relating x, y, z and write down the augmented matrix for those linear equations.

(b) By performing elementary row operations, put the augmented matrix into **reduced** row echelon form.

(c) Determine whether the equation system is consistent or inconsistent.

(d) Determine the solution set.

Required Knowledge: **Gauss-elimination, row operations, row echelon form, notions of lead/free variables.**

Solution:

1a: If x, y, and z be number of, respectively, type **A**, **B**, and **C**, products that produced each day, for assembling we get the equation

$$20x + 24y + 12z = 3120,$$

for testing

$$4x + 5y + 3z = 680,$$

and for packaging

$$4x + 4y + 3z = 640.$$

Augmented matrix for these equations, then, is given by:

$$\left[\begin{array}{ccc|c} 20 & 24 & 12 & 3120 \\ 4 & 5 & 3 & 680 \\ 4 & 4 & 3 & 640 \end{array}\right].$$

1b: By apylying elementary row operations, we can proceed as follows:

$$\left[\begin{array}{ccc|c} 20 & 24 & 12 & 3120 \\ 4 & 5 & 3 & 680 \\ 4 & 4 & 3 & 640 \end{array}\right] \xrightarrow[\textcircled{1} = \textcircled{1} - 5\cdot\textcircled{1}]{\textcircled{2} = \textcircled{2} - \textcircled{3}} \left[\begin{array}{ccc|c} 0 & 4 & -3 & -80 \\ 0 & 1 & 0 & 40 \\ 4 & 4 & 3 & 640 \end{array}\right] \xrightarrow{\textcircled{1} = \textcircled{1} - 4\cdot\textcircled{2}} \left[\begin{array}{ccc|c} 0 & 0 & -3 & -240 \\ 0 & 1 & 0 & 40 \\ 4 & 4 & 3 & 640 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 0 & 0 & -3 & -240 \\ 0 & 1 & 0 & 40 \\ 4 & 4 & 3 & 640 \end{array}\right] \xrightarrow{\textcircled{1} = -\frac{1}{3}\cdot\textcircled{1}} \left[\begin{array}{ccc|c} 0 & 0 & 1 & 80 \\ 0 & 1 & 0 & 40 \\ 4 & 4 & 3 & 640 \end{array}\right] \xrightarrow{\textcircled{3} = \textcircled{3} - 3\cdot\textcircled{1}} \left[\begin{array}{ccc|c} 0 & 0 & 1 & 80 \\ 0 & 1 & 0 & 40 \\ 4 & 4 & 0 & 400 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 0 & 0 & 1 & 80 \\ 0 & 1 & 0 & 40 \\ 4 & 4 & 0 & 400 \end{array}\right] \xrightarrow{\textcircled{3} = \textcircled{3} - 4 \cdot \textcircled{2}} \left[\begin{array}{ccc|c} 0 & 0 & 1 & 80 \\ 0 & 1 & 0 & 40 \\ 4 & 0 & 0 & 240 \end{array}\right] \xrightarrow{\textcircled{3} = \frac{1}{4} \cdot \textcircled{3}} \left[\begin{array}{ccc|c} 0 & 0 & 1 & 80 \\ 0 & 1 & 0 & 40 \\ 1 & 0 & 0 & 60 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 0 & 0 & 1 & 80 \\ 0 & 1 & 0 & 40 \\ 1 & 0 & 0 & 60 \end{array}\right] \xrightarrow[]{\substack{\textcircled{3} = \textcircled{1} \\ \textcircled{1} = \textcircled{3}}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 60 \\ 0 & 1 & 0 & 40 \\ 0 & 0 & 1 & 80 \end{array}\right].$$

1c: From the reduced row echelon form, we see that the system is consistent.

1d: The solution set consists of a single element $(x, y, z) = (60, 40, 80)$.

Let a, b, c, and d be real numbers. Find the determinant of the matrix

$$\begin{bmatrix} a^2-1 & a-1 & a-1 & a-1 \\ a+1 & b^2 & b & b \\ a+1 & b+2 & c^2+1 & c+1 \\ a+1 & b+2 & c+3 & d^2+2 \end{bmatrix}.$$

REQUIRED KNOWLEDGE: **Determinants, effects of elementary row operations on determinants.**

SOLUTION:

Let

$$M = \begin{bmatrix} a^2-1 & a-1 & a-1 & a-1 \\ a+1 & b^2 & b & b \\ a+1 & b+2 & c^2+1 & c+1 \\ a+1 & b+2 & c+3 & d^2+2 \end{bmatrix}.$$

We first observe that the first row of the matrix M is a multiple of $a-1$. This leads to

$$\det(M) = (a-1)\begin{vmatrix} a+1 & 1 & 1 & 1 \\ a+1 & b^2 & b & b \\ a+1 & b+2 & c^2+1 & c+1 \\ a+1 & b+2 & c+3 & d^2+2 \end{vmatrix}.$$

Now, we see that the first column of the matrix on the right is a multiple of $a+1$. Therefore, we get

$$\det(M) = (a-1)(a+1)\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & b^2 & b & b \\ 1 & b+2 & c^2+1 & c+1 \\ 1 & b+2 & c+3 & d^2+2 \end{vmatrix}.$$

By subtracting the first row from the second, third, and fourth, we obtain

$$\det(M) = (a^2-1)\begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & b^2-1 & b-1 & b-1 \\ 0 & b+1 & c^2 & c \\ 0 & b+1 & c+2 & d^2+1 \end{vmatrix}.$$

since type III elementary row operations do not change the determinant. By expanding along the first column, we can then write:

$$\det(M) = (a^2-1)\begin{vmatrix} b^2-1 & b-1 & b-1 \\ b+1 & c^2 & c \\ b+1 & c+2 & d^2+1 \end{vmatrix}.$$

By observing that the first row of the matrix on the right hand side is a multiple of $b-1$ and the first column of $b+1$, we see that

$$\det(M) = (a^2-1)(b^2-1)\begin{vmatrix} 1 & 1 & 1 \\ 1 & c^2 & c \\ 1 & c+2 & d^2+1 \end{vmatrix}.$$

Now, we subtract the first row from the second and third:

$$\det(M) = (a^2-1)(b^2-1)\begin{vmatrix} 1 & 1 & 1 \\ 0 & c^2-1 & c-1 \\ 0 & c+1 & d^2 \end{vmatrix}.$$

This leads to

$$\det(M) = (a^2-1)(b^2-1)\begin{vmatrix} c^2-1 & c-1 \\ c+1 & d^2 \end{vmatrix}.$$

Similar to before, we get

$$\det(M) = (a^2-1)(b^2-1)(c^2-1)\begin{vmatrix} 1 & 1 \\ 1 & d^2 \end{vmatrix}.$$

Finally, we obtain

$$\det(M) = (a^2-1)(b^2-1)(c^2-1)(d^2-1).$$

Let A be $n \times n$ nonsingular matrix. Consider the matrix

$$M = \begin{bmatrix} A & A & 0 \\ A & I & A \\ 0 & A & A \end{bmatrix}$$

where I and 0, respectively, denote the $n \times n$ identity and zero matrices.

(a) Show that the matrix M is nonsingular if and only if $I - 2A$ is nonsingular.

(b) Let $A = I$ and find the inverse of M.

REQUIRED KNOWLEDGE: **Partitioned matrices, nonsingular matrices, and inverse.**

SOLUTION:

3a: 'only if': Suppose that the matrix M is nonsingular. Let x be such that $(I - 2A)x = 0$. Note that

$$M \begin{bmatrix} -x \\ x \\ -x \end{bmatrix} = \begin{bmatrix} A & A & 0 \\ A & I & A \\ 0 & A & A \end{bmatrix} \begin{bmatrix} -x \\ x \\ -x \end{bmatrix} = \begin{bmatrix} -Ax + Ax \\ -Ax + x - Ax \\ Ax - Ax \end{bmatrix} = 0.$$

Since M is nonsingular, we see that $x = 0$. Hence, $I - 2A$ is nonsingular.

'if': Suppose that $I - 2A$ is nonsingular. Let x, y, z be such that

$$M \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0.$$

This means that

$$0 = \begin{bmatrix} A & A & 0 \\ A & I & A \\ 0 & A & A \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} A(x+y) \\ y + A(x+z) \\ A(y+z) \end{bmatrix}.$$

Since A is nonsingular, it follows from $A(x+y) = 0 = A(y+z)$ that $x = -y = z$. Then, it follows from $y + A(x+z) = 0$ that $y - 2Ay = 0$ and hence $(I - 2A)y = 0$. Since $I - 2A$ is nonsingular, we see that $y = 0$. As $x = z = -y$, we can conclude that $x = y = z = 0$. Consequently, M is nonsingular.

3b: In order to find the inverse of M when $A = I$, we will apply elementary row operations on the matrix

$$\begin{bmatrix} M \mid I_{3n} \end{bmatrix} = \left[\begin{array}{ccc|ccc} I & I & 0 & I & 0 & 0 \\ I & I & I & 0 & I & 0 \\ 0 & I & I & 0 & 0 & I \end{array}\right].$$

First we subtract the first row block from the second. This leads to

$$\left[\begin{array}{ccc|ccc} I & I & 0 & I & 0 & 0 \\ 0 & 0 & I & -I & I & 0 \\ 0 & I & I & 0 & 0 & I \end{array}\right].$$

Then, we subtract the second row block from the third:

$$\left[\begin{array}{ccc|ccc} I & I & 0 & I & 0 & 0 \\ 0 & 0 & I & -I & I & 0 \\ 0 & I & 0 & I & -I & I \end{array}\right].$$

Now, we subtract the second row block from the first:

$$\left[\begin{array}{ccc|ccc} I & 0 & 0 & 0 & I & -I \\ 0 & 0 & I & -I & I & 0 \\ 0 & I & 0 & I & -I & I \end{array}\right].$$

Finally, we interchange the second and third row blocks:

$$\left[\begin{array}{ccc|ccc} I & 0 & 0 & 0 & I & -I \\ 0 & I & 0 & I & -I & I \\ 0 & 0 & I & -I & I & 0 \end{array}\right].$$

Therefore, we obtain:

$$\begin{bmatrix} I & I & 0 \\ I & I & I \\ 0 & I & I \end{bmatrix}^{-1} = \begin{bmatrix} 0 & I & -I \\ I & -I & I \\ -I & I & 0 \end{bmatrix}.$$

Consider the vector space P_4. Let

$$S = \{p \in P_4 \mid p(0) = p(1) = p(2)\}.$$

(a) Show that S is a subspace.

(b) Find a basis for S and determine its dimension.

REQUIRED KNOWLEDGE: **Subspaces, basis, dimension.**

SOLUTION:

4a: First, we note that the zero polynomial belongs to S. As such, S is nonempty. Let α be a scalar and $p \in S$. Recall that

$$(\alpha p)(x) = \alpha p(x). \tag{1}$$

Define

$$q(x) = (\alpha p)(x). \tag{2}$$

Note that $q(0) \overset{(2)}{=} (\alpha p)(0) \overset{(1)}{=} \alpha p(0) \overset{p \in S}{=} \alpha p(1) \overset{(1)}{=} (\alpha p)(1) \overset{(2)}{=} q(1)$. Similarly, we have $q(1) \overset{(2)}{=} (\alpha p)(1) \overset{(1)}{=} \alpha p(1) \overset{p \in S}{=} \alpha p(2) \overset{(1)}{=} (\alpha p)(2) \overset{(2)}{=} q(2)$. Therefore, $q(0) = q(1) = q(2)$. As such, q belongs to S and hence S is closed under scalar multiplication.

Let $p, q \in S$. Recall that

$$(p + q)(x) = p(x) + q(x). \tag{3}$$

Define

$$r(x) = (p + q)(x). \tag{4}$$

Note that $r(0) \overset{(4)}{=} (p + q)(0) \overset{(3)}{=} p(0) + q(0) \overset{p,q \in S}{=} p(1) + q(1) \overset{(3)}{=} (p + q)(1) \overset{(4)}{=} r(1)$. Similarly, we have $r(1) \overset{(4)}{=} (p + q)(1) \overset{(3)}{=} p(1) + q(1) \overset{p,q \in S}{=} p(2) + q(2) \overset{(3)}{=} (p + q)(2) \overset{(4)}{=} r(2)$. . Therefore, $r(0) = r(1) = r(2)$. As such, r belongs to S and hence S is closed under vector addition.

Consequently, S is a subspace.

4b: Let $p \in P_4$ where $p(x) = ax^3 + bx^2 + cx + d$. Observe that $p \in S$ if and only if $p(0) = p(1) = p(2)$, that is if and only if

$$d = a + b + c + d = 8a + 4b + 2c + d.$$

Therefore, $p \in S$ if and only if $d = a+b+c+d$ and $a+b+c+d = 8a+4b+2c+d$. This means that $p \in S$ if and only if $a + b + c = 0$ or $7a + 3b + c = 0$. By subtracting the first one from the second, we obtain $p \in S$ if and only if $a+b+c = 0$ or $6a+2b = 0$. Therefore, we see that $p \in S$ if and only if $b = -3a$ and $c = 2a$. In other words, $p \in S$ if and only if $p(x) = ax^3 - 3ax^2 + 2ax + d$ for some real number a and d. This means that every $p \in S$ can be written as $p(x) = a(x^3 - 3x^2 + 2x) + d$ for some real number a and d. In other words, the set $\{x^3 - 3x^2 + 2x, 1\}$ is a spanning set for S. Since the polynomials $x^3 - 3x^2 + 2x$ and 1 are linearly independent, we can conclude that $\{x^3 - 3x^2 + 2x, 1\}$ is a basis for S. Therefore, the dimension of S is 2.